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CONTENTS

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SOME INFORMATION PROCESSES FOR ARRAYS OF SENSORS

INTRODUCTION

In much of the work done using arrays of sensors it is assumed that the energy field can be represented adequately by a finite set of discrete sample values taken at points in space [1-10] and time [11-17]. Representing a field over an interval of time by a single, multidimensional array of data permits formulations of the sensor-array problem that are well suited to use of known mathematical techniques on existing mathematical machines. We consider here some resulting transformations and analysis procedures that can provide operationally useful information from operational acoustic arrays. In particular we discuss (a) multidimensional generalized transforms that map arrays of data into arrays of coefficients of basis functions and (b) formal analysis procedures that operate on the arrays of coefficients to provide estimates of desired field or source parameters. The data arrays may result from nonuniformly spaced sample points [18-21] in space and time. The basis functions are exemplified by exponentials in wavenumber-frequency space, and for that case the analysis may be applied to the array of coefficients to provide estimates of the spatial and temporal frequency components of the field.

The use of multidimensional Fourier transforms to estimate the properties of seismic, meteorologic, and acoustic fields is well known [22-26]. There are some particular advantages to their use on the acoustic fields arising from distant periodic acoustic sources, that is, fields composed of periodic plane waves. The use of general complex exponentials and of certain non-Fourier basis functions for representing selected classes of waveforms is also well known [27-32]. Some of these techniques and some related interpolation and approximation procedures are developed further for application to multidimensional arrays of nonuniformly spaced acoustic sensors. Analysis-of-variance techniques are then applied in the transform domain to obtain the desired estimates. A review of some previous work on the representation of continuous functions by sets of discrete values, that is, sampling and interpolation theory, is given in Appendix A.

Work by Iyer, Berg, and others [22-24] on seismic array processing by integral transformations and work by Andrews, Oppenheim, and others [33-36] on image processing was taken as a point of departure in studying multidimensional transformations for sampled data from arrays of sensors. The aliasing and side-lobe structures for Fourier transforms of two-dimensional arrays of data (time samples from one-dimensional arrays of sensors), and the affect of conventional smoothing (hamming, hanning, etc.) and of some other forms of smoothing on two-dimensional transformed data, were investigated. A computer program was developed to simulate an environment and an operational acoustic system, including estimating the parameters of signal components (i. e., estimating the location, frequency, and amplitude of each signal-component source).

These simulation studies, when applied to a model of one important Navy problem, showed that more general transformation procedures were needed if full advantage was to be taken, in real systems, of actual array characteristics and of the wave-equation constraints on space and time data from acoustic fields. In particular, when high performance must be achieved over several octaves of frequency, adequate sampling of adequate apertures in space and time becomes prohibitively costly. Conventional array processes lack effective synthesis procedures and generally lack desired uniformity of system performance over the expected range of environmental situations. Most actual synthesis procedures for space-time processing, those of Wiener, Bryn, Mermoz, Widrow, and others [23, 37-40], have seemed unattractive for certain important Navy acoustic problems, both because they assume either unnecessarily broad classes of signals or exactly known signals and because they do not lead to effective space and time sampling procedures (i. e., array design and sample spacing).

A promising approach, related both to work of Huggins et al. [27, 41, 42] and to much fundamental work in mathematical analysis [see, for example, Refs. 43, 44, and 45], is that of transformations using carefully selected basis sets (sets of multidimensional functions or sets of number-arrays) by which a desired class of multidimensional signals (acoustic fields) can be adequately and efficiently represented. As we are concerned only with finite intervals of space and time, with band-limited signals and noise, and with energy fields constrained by the acoustic wave equation and by other known source and medium constraints, the general representational problem may be simplified and narrowed to manageable size without doing violence to its relation to the real world. All of the mathematics is finite and discrete (and can be expressed interchangeably in mathematical language or in a suitable computer language). In particular, if we actually have to deal with only a moderate number (perhaps a few tens of thousands) of different signals, and if this number is a sufficiently small fraction of the total number of different signals that are allowed and resolved by the bandwidth and the space and time aperture of our system, we may reasonably take the signals themselves (or rather, the set of space and time samples of the signals) as our basis set. That is, if the number of different signals is small enough, it may be reasonable to provide analysis that is the equivalent of multidimensional matched filtering for each different signal [46].

In general the multidimensional transformation matrix for an arbitrary sample-point spacing and an arbitrary basis set cannot be factored to as great an extent as in fast-Fourier-transform processing of uniformly spaced samples. The conditions under which factoring of the matrix is maximized and the question of whether acceptable constraints exist on array design that provide a highly factorable matrix (i.e., that reduce processing operations by approximately $\log_2 m/m$) are being investigated. Where uniform sampling is done in one dimension of a multidimensional sample space, matrix factoring is possible (e.g., for an m-by-n sample set, if the m samples are uniformly spaced and the n samples are not uniformly spaced, the number of operations can approach $(m \log_2 m)\, n^2$. Thus if m is very large compared to n, a very efficient transformation is possible even with nonuniform spacing of the n samples). Other questions of weighting or smoothing criteria for transformations of nonuniformly spaced samples, of aliasing effects, and of sensitivity of the transformations to sample location errors, are being investigated.

By this transformation approach the entire field as seen by the sensors is represented by a single, multidimensional array of numbers (e. g., an array of complex numbers in wavenumber-frequency space). Established statistical testing procedures can be adapted for estimating the signal parameters. Initial computer simulation runs applying analysis of variance to simulated eight-element and 16-element acoustic arrays have been made. This technique yields estimates of the effects of physically significant subsets of the transformed data. The result is that expected of an optimum receiver, since the F test used can be derived by maximum-likelihood methods [47]. Further work is being done on modified analysis-of-variance procedures that will be more effective in the presence of highly nonisotropic and nonwhite noise fields. All of these test programs, as well as transformation programs, are kept in a form that can be applied to recorded field data from operational acoustic arrays as well as to the simulated acoustic fields that are used in developing the tests.

In summary, as a step toward the effective application of advanced programable digital machinery to the outputs of the sensors of large, wide-band, operational acoustic arrays, we aim for these capabilities:

1. Procedures for transforming sets of values at arbitrary (or nearly arbitrary) points in an energy field into a representation in an arbitrary (or nearly arbitrary) estimation space. That is, procedures for expanding a finite set of discrete data, representing a field at selected points in space and time, on a selected finite set of multidimensional basis functions.

2. Procedures for selecting sampling points in space and time and for selecting basis sets for signal representation, in terms of the expected characteristics of the acoustic field and in terms of the kinds of information wanted from the field.

3. Procedures for estimating desired parameters of acoustic fields (or acoustic sources) from the representation of the field on the selected estimation space (or basis set).

In this report we discuss some work on the first and the third of these. Generalized multidimensional transformations that can represent data from nonuniformly spaced time samples from nonuniformly spaced array elements, on a large class of sets of basis functions, are discussed in the next section. Analysis-of-variance processes for estimating the parameters of signals from their representations on the selected set of basis functions are discussed in the third section.

The familiar wavenumber-frequency (Fourier) transformation that is used on seismic-array data, together with the various statistical tests applied to the transformed data [23, 24, 48], is an example of the general approach considered here. We begin with a more abstract formulation of the array problem, because this facilitates consideration of some other concepts and techniques which appear promising for our application - that is, for a study of transformation and estimation techniques for real-time application to large operational acoustic arrays, using computing machinery forecast for about the period 1973-1978.

INTERPOLATION AND TRANSFORMATIONS

Finite series consisting of the linear combination of a finite number of basis functions have been used extensively to approximate continuous functions which have been sampled on a finite range of their independent variables. When the set of basis functions used is orthogonal on the sample points, these finite series exhibit many properties of infinite series and functional transformations [49]. As a result of this fact and as a result of simplification of calculations resulting from orthogonality of the basis functions, most of the work on finite series has been concentrated on the use of orthogonal basis functions. A desired representation of an acoustic field which has been sampled at unequally spaced intervals by sensors in an acoustic array may not result in orthogonality of the basis set.

In the discussion which follows, quite general expressions are derived for approximations based on nearly arbitrary selections of basis functions and sample points. These results are applied to processing of information sampled by acoustic arrays

Theory

Following notation similar to that used by Young and Huggins [29], a function $f(x)$ may be approximated over a finite range L of its independent variables by a finite sum of n basis functions $S_k(x)$ by

$$f(x) \approx \sum_{k=1}^{n} c_k S_k(x). \tag{1}$$

In this expression $f(x)$, c_k, and $S_k(x)$ may be real or complex. The quantity x denotes a vector, the components of which are the independent variables. Sampling $f(x)$ at the m points $x_j\ (j = 1, 2, \ldots, m)$ yields the m approximations

$$f(x_j) \approx \sum_{k=1}^{n} c_k S_k(x_j), \quad j = 1, 2, \ldots, m. \tag{2}$$

If the number of sampling points equals the number of basis functions $(m = n)$, the approximation sign in (2) may be replaced by an equal sign to yield n equations in the n c_k's which, if independent, may be solved for the c_k's. In this case the approximation of the continuous function in (1) becomes exact at the sampling points and the finite series becomes an interpolation between the sampled points. If the number of sampled points exceeds the number of basis functions $(m > n)$, the approximations (2) may be solved for the c_k's subject to some desirable constraint, e. g. , least-square error; $m = n$ is a special case of $m \geq n$ for which the error of representation of the function at the sample points is zero.

The differences between the left and right sides of (2) represent the errors in approximating $f(x)$ at the sampling points. Following the approach taken in infinite series representations the c_k's will be determined to minimize the sum of the squares of these error terms. Representing this sum by e^2,

$$e^2 = \sum_{j=1}^{m} \left| f(x_j) - \sum_{k=1}^{n} c_k S_k(x_j) \right|^2. \tag{3}$$

To minimize e^2 Eq. (3) will be differentiated with respect to the real and imaginary parts of each c_k and each derivative will be set equal to zero. This will result in n equations from the derivatives with respect to the real parts and n equations from the derivatives with respect to the imaginary parts. If these $2n$ equations are independent, they may be solved for the n complex values of c_k. Let a_k be the real part and b_k be the imaginary part of c_k:

$$c_k = a_k + ib_k.$$

Differentiating with respect to a_s and b_s yields

$$\frac{\partial e^2}{\partial a_s} = -\sum_{j=1}^{m} \left\{ S_s(x_j) \left[\overline{f(x_j)} - \sum_{k=1}^{n} \overline{c_k}\, \overline{S_k(x_j)} \right] + \overline{S_s(x_j)} \left[f(x_j) - \sum_{k=1}^{n} c_k S_k(x_j) \right] \right\} = 0 \tag{4}$$

and

$$\frac{\partial e^2}{\partial b_s} = -i \sum_{j=1}^{m} \left\{ S_s(x_j) \left[\overline{f(x_j)} - \sum_{k=1}^{n} \overline{c_k}\,\overline{S_k(x_j)} \right] \right.$$

$$\left. - \overline{S_s(x_j)} \left[f(x_j) - \sum_{k=1}^{n} c_k S_k(x_j) \right] \right\} = 0, \tag{5}$$

where the overbar denotes the complex conjugate. The $2n$ real equations (4) and (5) may be combined into n complex equations by setting

$$\frac{\partial e^2}{\partial a_s} + i \frac{\partial e^2}{\partial b_s} = 0, \tag{6}$$

which after simplification gives

$$\sum_{j=1}^{m} \overline{S_s(x_j)} \sum_{k=1}^{n} c_k S_k(x_j) - \sum_{j=1}^{m} \overline{S_s(x_j)}\, f(x_j) = 0\,. \tag{7}$$

Since the range of the sums in Eq. (7) are finite, the order of summation may be interchanged to yield

$$\sum_{k=1}^{n} c_k \sum_{j=1}^{m} \overline{S_s(x_j)}\, S_k(x_j) = \sum_{j=1}^{m} \overline{S_s(x_j)}\, f(x_j)\,. \tag{8}$$

Equations (8) represent a system of n equations in the n unknowns c_k which, if they are linearly independent, may be solved for the c_k's.

The conditions in Eqs. (4) and (5) which lead to Eq. (8) insure that the solutions of Eq. (8) will lead to a stationary point in e^2. As e^2 is a non negative quadratic form in the c_k's, this stationary point is assumed to be a minimum.

Examination of Eq. (8) indicates that considerable simplification will result from introduction of matrix notation. The m-by-n matrix P will be defined by

$$P = [P_{jk}] = [S_k(x_j)]. \tag{9}$$

We also introduce the vectors f and c:

$$f = [f_1, \ldots, f_m]' = [f(x_1), \ldots, f(x_m)]',$$

$$c = [c_1, \ldots, c_n]'.$$

In terms of these newly defined quantities, Eq. (8) may be written

$$P^{\dagger}Pc = P^{\dagger}f, \tag{10}$$

where $P^{\dagger}$ denotes the Hermitian conjugate of P, the n-by-m matrix which is the complex conjugate of the transpose of P: $P^{\dagger} \equiv [\overline{P_{kj}}]$. If the n-by-n matrix $(P^{\dagger}P)$ is nonsingular, its inverse exists, and we have

$$c = (P^{\dagger}P)^{-1} P^{\dagger}f. \tag{11}$$

Equation (11) states that $P^{\dagger}$ operates on the m-dimensional vector to project it on to an n-dimensional space; the n-dimensional c is computed from this projection by the transformation $(P^{\dagger}P)^{-1}$. Equation (11) permits computation of the coefficients c_k of the n basis functions $S_k(x)$ in (1) to yield a best (least-squares) approximation of the m sample values $f(x_j)$ *subject to the limitation that the n-by-n matrix formed from the S_k sampled at the x_j, $(P^{\dagger}P)^{-1}$, be nonsingular.* This is the only limitation imposed on the basis functions or the sample points to validate Eq. (11).

Three special cases of Eq. (11) are important in the discussion which follows: (a) the number of sample points and basis functions are equal $(m = n)$, (b) the basis functions form an orthonormal set, and (c) $f(x)$ is one of the basis functions.

When $m = n$, the matrices P, P^{-1}, and $P^{\dagger}$ are square. In this case the condition that $P^{\dagger}P$ be nonsingular requires that P^{-1} and $P^{\dagger}$ be nonsingular. Invoking the matrix identity $(AB)^{-1} = B^{-1}A^{-1}$ permits Eq. (11) to be written

$$c = (P^{\dagger}P)^{-1} P^{\dagger}f = P^{-1}P^{\dagger\,-1}P^{\dagger}f = P^{-1}f. \tag{12}$$

This equation could have been obtained directly from (2) with the approximation sign replaced by an equal sign. It permits the finite series in (1) to be fitted exactly to sample points $f(x_j)$. Choice of the basis functions and the sample points is subject only to the limitation that the matrix P be nonsingular. Thus, for example, the spectrum of a continuous function $f(x)$ may be estimated from a set of samples $f(x_j)$ even though the sampling points are not equally spaced on x. Similarly the spectrum may be estimated at a set of nonharmonically related frequencies. It should be noted that P^{-1} depends only on the basis functions and the sample points and thus need not be recalculated for each new set of sampled data. The techniques of the fast Fourier transform [49] are applicable to the computation of Eq. (12) if the matrix P^{-1} is factorable but in general will not be as advantageous as in the case of harmonically related basis functions and equally spaced sampling points.

If the basis functions form an orthonormal (not necessarily complete) set, then

$$\sum_{j=1}^{m} \overline{S_k(x_j)}\; S_h(x_j) = \delta_{kh}, \tag{13}$$

i. e., $P^{\dagger}P = I$ where I is the n-by-n identity matrix. In this case Eq. (11) becomes $c = P^{\dagger}f$, or

$$c_k = \sum_{j=1}^{m} \overline{S_k(x_j)}\; f(x_j). \tag{14}$$

The expression is independent of the relative sizes of m and n; that is, it is valid for both exact fitting of the data at the sample points and for fitting in the least-squares sense. In other words the coefficients of the basis functions which form an orthonormal set are independent of which subset of the functions is used to form a least-squares fit to the data. A special case of this is the truncation of a Fourier series for smoothing experimental data [50].

If the function to be approximated is one of the basis functions, the vector f is the kth column of the matrix P. Thus $P^{\dagger}f$ is the kth column of the matrix $(P^{\dagger}P)$. By definition of the inverse $(P^{\dagger}P)^{-1}$ the kth element of $(P^{\dagger}P)^{-1}(P^{\dagger}f)$ will be unity and all other elements will be zero. This outlines the proof of the intuitive conclusion that, if $f(x)$ is one of the basis functions, approximation (1) will be a least-squares approximation to $f(x)$ if the coefficient of only the appropriate basis function is different from zero. This result will be of special interest in the discussion of processing signals from an acoustic array.

Interpolation and Smoothing

Expression (1) is the approximation of a function over a continuous interval of its independent variable. Equation (11) permits calculation of the coefficients c_k in (1) to make it a best (least-squares) approximation to the sampled data by using a matrix determined by sampling the basis functions at the sampling points. When the number of sampling points exceeds the number of basis functions $(m > n)$, Eq. (2) approximates the function at the sample points, in the least-squares sense, and (1) may be considered a smoothed representation of the continuous function $f(x)$. When the number of sampling points equals the number of basis functions $(m = n)$, (1) is exact at the sampling points x_j and provides interpolation between the sampling points.

To simplify the notation further we introduce the matrix $U = (P^{\dagger}P)^{-1} P^{\dagger}$. Then Eq. (11) may be written $c = Uf$, or

$$c_k = \sum_{j=1}^{m} U_{kj} f(x_j). \tag{15}$$

Substituting (15) in (1) yields

$$f(x) \approx \sum_{k=1}^{n} \sum_{j=1}^{m} U_{kj} f(x_j) S_k(x)$$

or

$$f(x) \approx \sum_{j=1}^{m} \left[\sum_{k=1}^{n} U_{kj} S_k(x) \right] f(x_j). \tag{16}$$

The inner sum in (16) is an interpolation function when $m = n$ and determines the contribution of each sample point to the continuous approximation of $f(x)$. It is a smoothing function when $m > n$. Let

$$I(x, x_j) = \sum_{k=1}^{n} U_{kj} S_k(x), \quad j = 1, 2, \ldots, m, \tag{17}$$

represent these interpolation (or smoothing) functions. The exact forms of the I's will depend on the choice of the basis functions and the sampling points and in general will be different for each x_j. Some examples of the analogous infinite case are given in Appendix A, and the finite case is illustrated in Appendix B.

Spectrum Estimates - Discrete Fourier Transforms

For acoustic signal detection and parameter estimation it is desirable to use the coefficients c_k in (1) rather than the continuous approximation represented by that equation. It is customary to speak of the c_k's as a "transformed" representation of the $f(x_j)$'s, the transformation being carried out by the matrix $(P^\dagger P)^{-1} P^\dagger$ in Eq. (11). The exact nature of the transformation and the appropriate interpretation of the result will depend on the basis functions chosen and on selection of the sample points x_j.

The most widely used transformations of this type involve imaginary exponentials for basis functions:

$$S_k(x) = e^{2\pi i f_k x}, \tag{18}$$

where the f_k are chosen to be equally spaced in the "frequency" domain and the x_j are equally spaced on x. These transformations have many properties similar to the conventional Fourier transform [49 - 52] and have therefore become known as the discrete Fourier transform (DFT).

If the function $f(x)$ is band limited and the sample point spacing satisfies the Nyquist criterion, and if $f(x)$ is periodic on the interval over which the samples are taken, i. e., $f(x) = f(x + L)$, the DFT will yield precisely the same information as the conventional Fourier transform. Of considerable interest is the interpertation of the DFT when these criteria are not satisfied.

Because of the parallels with computation of the Fourier transform, calculation of the c_k's for the case of equally spaced f_k and x_j is usually referred to as determination of the complex spectrum of $f(x)$. Estimation of the complex spectrum from the c_k's has received considerable attention in the literature [53, 54]. Errors from the use of the DFT to estimate the spectrum arise from two sources: the finite range of x over which $f(x)$ is sampled and the finite number of samples taken within this range. As indicated, these limitations do not give rise to errors when the function $f(x)$ is band limited and is periodic over the interval on which it is sampled. For more general functions the first limitation (finite interval of x) results in the spectrum being sampled by a finite window, which may extend over a considerable range of frequencies. The second limitation (the finite number of samples within the interval) results in aliasing of the spectrum. Rather than treat these phenomena as separate this discussion will show that they both may be considered as direct consequences of the way in which the c_k's are generated. We will first consider the interpretation for a single imaginary exponential and then treat a more general $f(x)$ as the superposition of such simple functions.

First consider the case for which $f(x)$ is the single exponential

$$f(x) = e^{2\pi i f x}.$$

The usual computation of the DFT corresponds to selection of basis functions and sample points:

$$S_k(x) = e^{2\pi i(k-1)x/L)}; \; x_j = (j-1)\,L/n, \tag{19}$$

where L is the interval over which $f(x)$ is sampled and n is the number of sample points within the interval. Using the orthogonality of the $S_k(x_j)$, the coefficients of the DFT may easily be shown to be

$$c_k = \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i(fL-k+1)(j-1)/n}. \tag{20}$$

When fL is an integer, $f(x)$ is one of the basis functions and we have, in accord with the previous discussion, $c_{fL+1} = 1$, and $c_k = 0$ if $k \neq fL+1$. When fL is not an integer, Eq. (20) generally yields nonzero values for all c_k, which may be interpreted as spreading or leakage caused by the window referred to earlier.

An interesting interpretation of Eq. (20) is obtained by considering the interpolation function which arises from the choice of basis functions and sample points in Eq. (19). From Eq. (17) that interpolation function is

$$I(x, x_j) = \frac{1}{n}\sum_{k=1}^{n} e^{2\pi i(nx/L-j+1)(k-1)/n}. \tag{21}$$

This expression is exactly the same form as that for the DFT in Eq. (20) with nx/L playing the role of fL and the roles of j and k being reversed. This symmetry arises from the orthogonality of the $S_k(x)$ and the symmetrical appearance of k and j in both Eq. (19) and Eq. (20) and *does not hold for other choices of the basis functions and sample points.* For this case (basis functions which are equally spaced, imaginary exponentials, and equally spaced sample points) the spectral window has just the same form as the interpolation function based on the same choice of basis functions and sample points.

If $f(x)$ is of the class $L_2(-\infty, \infty)$, it may be expressed as a superposition of imaginary exponentials by its Fourier integral:

$$f(x) = \int_{-\infty}^{\infty} F(f)\; e^{2\pi i f x}\, df\,, \tag{22}$$

where $F(f)$ is the complex spectrum or Fourier transform of $f(x)$. Again using the orthogonality of the $S_k(x_j)$,

$$\begin{aligned} c_k &= \int_{-\infty}^{\infty} F(f)\left[\frac{1}{n}\sum_{j=1}^{n} e^{2\pi i(FL-k+1)(j-1)/n}\right] df \\ &= \int_{-\infty}^{\infty} F(f)\, w(f)\, df\,, \end{aligned} \tag{23}$$

where the weighting function $w(f)$ is the spectral window applied to $F(f)$ in calculating the DFT. The utility of the c_k's for estimating $F(f)$ stems from the fact that $w(f)$ (which as we have seen has the form of the corresponding interpolation function in

Eq. (21) has a maximum value at $fL = k - 1$ and is smaller elsewhere. As the number of sample points tends toward infinity, we have

$$c_k \rightarrow \int_{-\infty}^{\infty} F(f) \left[e^{\pi i (FL-k+1)} \frac{\sin \pi (fL - k + 1)}{\pi (fL - k + 1)} \right] df \tag{24}$$

The exponential term arises because the sample points x are not symmetrically disposed about $x = 0$, and the sin x/x term is the actual shape of the spectral window.

It can be seen that, as the interval L over which $f(x)$ is sampled (the aperture) increases, the width of the window decreases. It can also be seen that its shape is independent of k. Thus each c_k arises from application of the same window, centered about a different frequency $f = (k - 1)/L$.

It can be seen from Eq. (23) that the window used in determining c_k is periodic on the interval n/L. Thus $w(f)$ has peaks at

$$f = \frac{k - 1 + jn}{L}, \quad j = 0, \pm 1, \pm 2, \ldots . \tag{25}$$

If the c_k is considered to represent $F(f)$ in the vicinity of the peaks of $w(f)$, it is seen that each c_k may contain contributions from $F(f)$ at several frequencies. This phenomenon is the spectral aliasing mentioned earlier. Even if the Nyquist criterion is satisfied, i. e., $F(f) = 0$ for $|f| > n/2L$, some aliasing can occur as a result of the finite width of the window.

When f_k or x_j are unequally spaced and $f(x)$ is a single imaginary exponential, the coefficients c_k may be written, using Eq. (15), as

$$c_k = \sum_{j=1}^{m} U_{kj} \, e^{2\pi i f x_j}. \tag{26}$$

As in the previous discussion Eq. (26) when considered a function of f is the weighting function applied to $F(f)$ in determining the approximate spectrum of $f(x)$. The symmetry between j and k is not present in this equation, and in general the form of the function depends on k; i. e., the weighting function differs for each coefficient c_k. If the sample points x_j are equally spaced (whether or not the f_k are), the c_k will be periodic in f on an interval of $n/2L$ and the preceding discussion of spectrum folding applies. When the x_j are not rationally related, the c_k will not be periodic but will still have a multiplicity of peaks. In general the peaks will have different magnitudes and shapes, indicating that the weighting applied to each alias will be different. There is no assurance that the peak at $f = (k - 1)/L$ will be the largest, as can be seen in Appendix B.

Power Spectrum and Total Power

The power density function PDF of $f(x)$ as expressed in Eq. (22) is usually given to be

$$W(f) = |F(f)|^2.$$

To the extent that c_k represents an approximation of $F(f)$, $|c_k|^2$ may be used as an approximation of $W(f)$

The total power is expressed as

$$\Pi = \int_{-\infty}^{\infty} W(f)\, df$$

or

$$\Pi = \lim_{X\to\infty} \frac{1}{2X} \int_{-X}^{X} |f(x)|^2\, dx\,. \tag{27}$$

A useful estimate of the total power is

$$\Pi \approx \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^2 = \frac{1}{n} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} c_k S_k(x_j) \right|^2,$$

which becomes

$$\Pi = \sum_{k=1}^{n} \sum_{h=1}^{n} c_k \overline{c_h} \left[\frac{1}{n} \sum_{j=1}^{n} S_k(x_j)\, \overline{S_h(x_j)} \right]. \tag{28}$$

When the basis functions are orthogonal on the sample points, this simplifies to

$$\Pi = \sum_{k=1}^{n} |c_k|^2, \tag{29}$$

which is parallel to Eq. (27). It is important to note that Eq. (29) does not hold for nonorthogonal basis functions.

Smoothed Spectral Estimates

Even when large numbers of samples and large apertures are used, c_k and $|c_k|^2$ have disadvantages for estimating the spectrum $F(f)$ and the power density function $W(f)$. One disadvantage, already mentioned, is that the weighting function $w(f)$ may not fall off sufficiently rapidly from its peak value to permit accurate estimation of $F(f)$ for those cases where $F(f)$ itself may have large peak values. That is, the leakage caused by the "side lobes" of the spectral window may cause a large peak in $F(f)$ to mask the true spectral levels at other frequencies. A second problem is that for many classes of $f(x)$ the c_k may not be a consistent estimator of $F(f)$ [54]. For these reasons, considerable attention has been given to the problem of improving the estimates, and in particular to estimation of the PDF. Some techniques involve operations on $f(x)$ directly, before transformation, and others operate on the c_k's after transformation. In whatever domain the smoothing operation is performed, it may be considered an attempt to improve the spectral window used to determine the c_k's.

Consider, for example, application of a weighting function applied in the x domain:

$$f'(x) = f(x)\ \theta(x). \tag{30}$$

The Fourier transform of $f'(x)$ is the convolution of $F(f)$ and $\Theta(f)$, the transform of $\theta(x)$:

$$F(f) = \int \Theta(f - \lambda)\ F(\lambda)\ d\lambda.$$

The coefficient c_k is thus seen from Eq. (23) to be

$$\begin{aligned} c_k &= \int F'(f) w(f)\ df = \iint w(f) \Theta(f - \lambda) F(\lambda)\ df d\lambda \\ &= \int F(f) \left[\int \Theta(\lambda)\ w(f + \lambda)\ d\lambda \right] df. \end{aligned} \tag{31}$$

The expression in brackets is a modified weighting function formed from $\Theta(f)$ and the original weighting function $w(f)$. Thus the new weighting function, or window, is formed from the old by a process which does not depend upon selection of the basis functions or sample points.

For some applications it is desirable to perform the smoothing operation in the f domain; for example, for n basis functions

$$c'_k = \sum_h a_h c_{k+h};\ \ c_{n+s} \equiv c_s,\ c_{-k} \equiv c_{n-k}. \tag{32}$$

If the operation is to be carried out frequently using the same smoothing function, it may be desirable to introduce a new matrix P', computed once and then used in place of P^{-1} in computing the c_k's. Thus

$$[P'_{jk}] = \sum_h a_h \left[P^{-1}_{j,\,k+h}\right]. \tag{33}$$

For the special case where the frequencies of the basis functions are equally spaced the spectrum of $\theta(x)$ is a sum of delta functions,

$$\Theta(f) = \sum_h a_h\ \delta(f - h),$$

and the corresponding weighting function to be applied in the x domain is

$$\theta(x_j) = \sum_h a_h\ e^{2\pi i h x_j / L}. \tag{34}$$

Most of the smoothing functions discussed in the literature are symmetrical about $h = 0$, in which case Eq. (34) becomes

$$\theta(x_j) = a_0 + \frac{1}{2} \sum a_h \cos\ (2\pi h x_j / L). \tag{35}$$

Note that these equations are applicable for arbitrarily spaced x_j.

Sampling and Transforming Acoustic Fields

A wave in a three-dimensional medium can be represented by

$$F(k_x, k_y, k_z, f) = \iiiint f(x, y, z, t)\, e^{-2\pi i(k_x x + k_y y + k_z z + ft)} dx dy dz dt, \tag{36}$$

where the k's are the wavenumber components in the direction of their respective space coordinate axes [24]. The wave along a line in the field can be represented by

$$F(k_x, f) = \iint f(x, t)\, e^{-2\pi i(k_x x + ft)} dx dt, \tag{37}$$

which is analogous to the Fourier transform representation of a picture in transform image coding [55]. Finite discrete processes (digital computers) can, of course, perform transformations on sampled data from acoustic arrays. The discrete Fourier transform analogous to the transform of Eq. (37) has been used in several simulation experiments, and one example of its use with an analysis of variance process is discussed in the next section.

The pressure ψ in an acoustic field in a homogeneous medium satisfies the wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{38}$$

Its solution can be expressed as single-frequency plane waves summed over all directions of travel and over all frequencies. This plane-wave expansion may be written

$$\psi(x, y, z, t) = \iiint \Psi(\phi, \theta, k)\, e^{ik[ct - \sin\theta(x\cos\phi + y\sin\phi) - z\cos\theta]} d\phi d\theta dk, \tag{39}$$

where ϕ and θ are the usual polar coordinates of each plane-wave component.

Many acoustic problems (for example detecting and locating specific sound sources) require estimating Ψ as a function of wavenumber and direction from samples of ψ taken over a limited region of space and over a finite time.

Let the coordinates of the h^{th} sensor be the components of the vector r_h, $h = 1, \ldots, m$, and let the output of each sensor be sampled at the times t_j, $j = 1, \ldots, n$. Then a data array consisting of $m \times n$ data points $\psi(r_h, t_j)$ results. Using the approach taken previously, some $N \le m \times n$ basis functions could be selected to approximate the field, and the techniques previously described could be applied to determining the coefficients of the approximation and interpreting the results as an estimation of the desired properties of the field. Instead we may choose m' basis functions which depend on space only and n' basis function which depend on time only. The approximation is the double sum over products of these basis functions:

$$\psi(r, t) \approx \sum_{p=1}^{m'} \sum_{q=1}^{n'} c_{pq} S_p(r) T_q(t). \tag{40}$$

Approximation (40) is the equivalent of (1). Following the procedure used previously, the error in approximating each of the values in the data array may be determined and the coefficients chosen to minimize the sum of the squares of these errors.

The result, equivalent to Eq. (7), is

$$\sum_{h=1}^{m}\sum_{j=1}^{n}\overline{S_u(r_h)}\,\overline{T_v(t_j)}\sum_{p=1}^{m'}\sum_{q=1}^{n'}c_{pq}S_p(r_h)T_q(t_j)=\sum_{h=1}^{m}\sum_{j=1}^{n}\overline{S_u(r_h)}\,\overline{T_v(t_j)}\,\psi(r_h,\ t_j)$$

or equivalently

$$\sum_{p=1}^{m'}\sum_{q=1}^{n'}\left[\sum_{h=1}^{m}\overline{S_u(r_h)}\,S_p(r_h)\right]c_{pq}\left[\sum_{j=1}^{n}\overline{T_v(t_j)}\,T_q(t_j)\right]=\sum_{h=1}^{m}\sum_{j=1}^{n}\overline{S_u(r_h)}\;\psi(r_h,t_j)\,\overline{T_v(t_j)}. \tag{41}$$

The matrices P and Q are defined in terms of the basis functions as

$$P\equiv[P_{hp}]\equiv S_p(r_h)\ \text{and}\ Q\equiv[Q_{qj}]\equiv T_q(t_j). \tag{42}$$

Those representing the array of coefficients c_{pq} and the data array are

$$c\equiv[c_{pq}]\ \text{and}\ \psi\equiv[\psi_{hj}]\equiv\psi(r_h,\ t_j). \tag{43}$$

In terms of these matrices Eq. (41) may be written

$$(P^{\dagger}P)\ c\ (QQ^{\dagger})=P^{\dagger}\ \psi Q^{\dagger}. \tag{44}$$

If the matrices $P^{\dagger}P$ and $QQ^{\dagger}$ are nonsingular (the only limitation on selection of basis functions and sample points), Eq. (44) may be solved for c as

$$c=(P^{\dagger}P)^{-1}\ P^{\dagger}\psi Q^{\dagger}(QQ^{\dagger})^{-1}. \tag{45}$$

Equation (45) is analogous to Eq. (11). Note that the order in which the space transform $(P^{\dagger}P)^{-1}\ P^{\dagger}$ and the time transform $Q^{\dagger}(QQ^{\dagger})^{-1}$ are applied is immaterial. We introduce the matrices U and V such that

$$U\equiv[U_{ph}]\equiv(P^{\dagger}P)^{-1}P^{\dagger}\ \text{and}\ V\equiv[V_{jq}]\equiv Q^{\dagger}(QQ^{\dagger})^{-1}. \tag{46}$$

The discussion of Eq. (11) involving equality of m and n, orthogonality of the basis function on the sample points etc. applies to Eq. (45). In particular, if $m=m'$, then $U=P^{-1}$ and if $n=n'$, then $V=Q^{-1}$. If the basis functions are orthonormal on their respective sample points, then $U=P^{\dagger}$ and $V=Q^{\dagger}$, which leads to

$$c_{pq}=\sum_{h=1}^{m}\sum_{j=1}^{n}\overline{S_p(r_h)}\,\overline{T_q(t_j)}\,\psi(r_h,\ t_j). \tag{47}$$

For arbitrary m and n the procedure which led to Eq. (17) now leads to the analogous interpolation function (or smoothing function)

$$I(r,\ r_h,\ t,\ t_j) = \sum_{p=1}^{m'} U_{ph} S_p(r) \sum_{q=1}^{n'} V_{jq} T_q(t)\ . \tag{48}$$

This is just the product of two interpolation functions of the type which has already been discussed.

To estimate $\Psi(\phi,\ \theta,\ k)$ in Eq. (39) we consider first the case where $\psi(r,\ t)$ is a single plane wave of a single frequency. Then

$$\Psi(\phi,\ \theta,\ k) = \delta(\phi-\phi_0)\ \delta(\theta-\theta_0)\ \delta(k-k_0)\ ,$$

and the pressure wave may be written

$$\psi(r,\ t) = e^{ik_0(ct-r\cdot N_0)}, \tag{49}$$

where N_0 is a unit vector in the direction of travel of the wave and

$$r\ \cdot N_0 \equiv (x\cos\phi_0 + y\sin\phi_0)\ \ \sin\theta_0 + z\ \cos\ \theta_0\ .$$

For this case the coefficients in (40) become

$$c_{pq} = \sum_{h=1}^{m} U_{ph}\ e^{-k_0 r_h\cdot N_0} \sum_{j=1}^{n} V_{jq}\ e^{ik_0 ct_j},$$

which is the product of two expressions similar to that in Eq. (26). For the plane-wave expansion in Eq. (39), the coefficients may be written

$$c_{pq} = \iiint w'\ (\phi,\theta,k)\, w''(k)\ \Psi(\phi,\theta,k)\ d\phi d\theta dk\ , \tag{50}$$

where the weighting functions w' and w'' are

$$w'(\phi,\ \theta,\ k) = \sum_{h=1}^{m} U_{ph}\ e^{-ikr_h\cdot N}$$

and

$$w''(k) = \sum_{j=1}^{n} V_{jq}\ e^{ikct_j}.$$

These expressions are similar to Eq. (26), and the discussion pertaining to that equation involving folding or aliasing, shape, etc. applies here also. The dependence on k is a result of the link between space and time variables which arises because the pressure wave must satisfy wave equation (38).

ANALYSIS OF VARIANCE

The analysis of variance is a statistical technique used to separate the influence of different sets of parameters on observed data and to estimate their effects. It was originally developed by R. A. Fisher in the 1920's for application to agricultural experiments. We apply it here to the two-dimensional discrete Fourier transform of time samples from a line array of sensors.

In the form suited for this wavenumber-frequency data, the "two-way layout," we consider K replications of I-by-J data matrix. Let y_{ijk} denote, the ij^{th} cell. We assume initially that

$$y_{ikj} = \eta_{ij} + e_{ijk}, \tag{51}$$

where the $\{\eta_{ij}\}$ are unknown constants and the $\{e_{ijk}\}$ are (initially) independently and identically distributed normal variates with zero mean.

In the original applications the $\{y_{ijk}\}$ may have been crop yields, with the rows corresponding to plant variety and the columns to fertilizer type. The experimenter wished to determine the main effects of rows (variety) averaged over all columns (fertilizer), the main effect of columns averaged over all rows, and the interactions, if any, between rows and columns. The interactions were of secondary importance, but it was nevertheless essential to estimate them, since the effect of a fertilizer on one variety of plant may not have been the same as that on another.

The technique can of course be applied to any data satisfying the assumptions. Richters [56] has applied it to the seismic discrimination problem. He was concerned with estimation of the effects of "nuisance" parameters (such as event magnitude and distance). Here the absence of interactions between the nuisance parameters greatly simplifies the discrimination problem, Shumway [57] has derived a general theory for using regression and analysis of variance in the frequency domain as a simultaneous estimation and detection technique for multivariate time series. His model is that of a "one-way layout," so the concept of interaction does not enter.

By contrast, in our application to the Fourier transform of space-time samples it is the interaction between wavenumber (rows) and frequency (columns) that is of primary importance, although the main effects of wavenumber and of frequency may have interesting physical interpretations as well.

Least Squares Estimates fo the Parameters

We sketch briefly the relevant definitions and theory following Scheffé [47] (see especially section 4. 3). Using a common notation, a dot replacing a subscript indicates an average over the missing subscript. For example,

$$y_{ij.} = \sum_{k=1}^{K} y_{ijk}/K \text{ denotes the } ij^{\text{th}} \text{ cell mean,}$$

$$y_{i..} = \sum_{j=1}^{J} \sum_{k=1}^{K} y_{ijk}/JK \text{ denotes the } i^{\text{th}} \text{ row mean,}$$

and

$$y_{...} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} y_{ijk}/IJK \text{ denotes the overall mean.}$$

Let $\mu = \eta_{..}$ represent the (unknown) mean of the two-dimensional spectral components η_{ij}. The main effects of the i^{th} wavenumber and of the j^{th} frequency are defined as

$$\alpha_i = \eta_{i.} - \mu, \tag{52}$$

and

$$\beta_j = \eta_{.j} - \mu, \tag{53}$$

respectively.

The interaction of wavenumber and frequency is defined as

$$\gamma_{ij} = \eta_{ij} - \alpha_i - \beta_j - \mu. \tag{54}$$

Because $\mu = \eta_{..}$, we have the "side conditions"

$$\alpha_. = \beta_. = \gamma_{i.} = \gamma_{.j} = 0, \ \forall_i, \ \forall_j. \tag{55}$$

The essence of the analysis of variance technique is the examination of inhomogeneities in the data by means of which we make estimates $\hat{\alpha}_i$, $\hat{\beta}_j$, $\hat{\gamma}_{ij}$, and $\hat{\eta}_{ij}$ of these unknown parameters.

The variates e_{ijk} in Eq. (51) represent the noise, which is defined as all unwanted effects. If we consider the variates to be the "error," the least-squares estimate of the parameters is clearly obtained by minimizing the "sum of squares,"

$$SS = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \eta_{ij})^2. \tag{56}$$

An estimate, which will be denoted by a circumflex, is obtained by calculus as in the previous section. We have

$$\hat{\eta}_{ij} = \sum_{k=1}^{K} y_{ijk} = y_{ij} \ . \tag{57}$$

Using Eq. (54) and Eq. (66) we can write

$$SS = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \alpha_i - \beta_j - \gamma_{ij} - \mu)^2. \tag{58}$$

Equations (55) and (58) yield the least-squares estimates:

$$\hat{\mu} = \frac{1}{IJK} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} y_{ijk} = y_{...}, \tag{59}$$

$$\hat{\alpha}_i = \frac{1}{JK} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \hat{\mu}) = y_{i..} - y_{...}, \tag{60}$$

$$\hat{\beta}_j = \frac{1}{IK} \sum_{i=1}^{I} \sum_{k=1}^{K} (y_{ijk} - \hat{\mu}) = y_{.j.} - y_{...}, \tag{61}$$

$$\hat{\gamma}_{ij} = \frac{1}{K} \sum_{k=1}^{K} (y_{ijk} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\mu}) = y_{ij.} - y_{i..} - y_{.j.} + y_{...}. \tag{62}$$

It is important to note that in deriving these estimates no use has been made of any assumptions of normality, zero mean, or equal variance. We have simply minimized the least-squares "error," so that when these estimates exist they are valid for any distribution of the $\{e_{ijk}\}$. If the $\{e_{ijk}\}$ are pairwise uncorrelated, have zero means, and have the same variance, then these are the unique linear unbiased estimates with minimum variance [47, 58, 59]. Tests of their statistical significance must of course depend on the distribution.

Tests of Hypotheses

The general I-by-J analysis of variance with K replications per cell is usually concerned with the testing of three hypotheses:

$$H_A\text{: all } \alpha_i = 0, \tag{63a}$$

$$H_B\text{: all } \beta_j = 0, \tag{63b}$$

$$H_{AB}\text{: all } \gamma_{ij} = 0. \tag{63c}$$

Substituting the least-squares estimate, Eq. (57), for η_{ij} in Eq. (56) we obtain the minimum sum of squares, or the "error sum of squares,"

$$SS_e = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - y_{ij.})^2. \tag{64}$$

If the $\{e_{ijk}\}$ are independently normally distributed, SS_e can be shown [47] to have a chi-square distribution with $IJ(K-1)$ degrees of freedom.

Under the hypothesis H_A: all $\alpha_i = 0$ the sum of squares to be minimized is, from Eq. (58),

$$SS_{\omega_A} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \beta_j - \gamma_{ij} - \mu)^2 .$$

Its minimum value, obtained by substituting the estimates given by Eqs. (59), (61), and (62), is

$$SS_{\min_A} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - y_{ij.} + y_{i..} - y_{...})^2$$

$$= SS_e + JK \sum_{i} (y_{i..} - y_{...})^2 .$$

The hypothesis sum of squares is defined as

$$SS_A = SS_{\min_A} - SS_e$$

$$= JK \sum_{i} (y_{i..} - y_{...})^2 . \tag{65}$$

Under the assumption of normality SS_A can be shown [47] to have a chi-square distribution with $I - 1$ degrees of freedom and to be independent of SS_e. Thus, under this assumption, the (likelihood-ratio) statistic

$$F_A = \frac{SS_A/(I-1)}{SS_e/IJ(K-1)} \tag{66}$$

has an F distribution with $\nu_1 = I - 1$ and $\nu_2 = IJ(K-1)$ degrees of freedom. If α is the false-alarm probability, we then have

$$P\left[F_A \le F_{\alpha;\ \nu_1,\ \nu_2}\right] = 1 - \alpha , \tag{67}$$

where $F_{\alpha;\ \nu_1,\ \nu_2}$ is the "upper α point" of the F distribution. If $F_A > F_{\alpha;\ \nu_1,\ \nu_2}$, the hypothesis H_A is rejected at a level of confidence of α.

Under the hypothesis H_B: all $\beta_j = 0$ we have similarly

$$SS_B = IK \sum_{j} (y_{.j.} - y_{...})^2 . \tag{68}$$

Under the normality assumptions

$$F_B = \frac{SS_B/(J-1)}{SS_e/IJ(K-1)} \tag{69}$$

has an F distribution with $\nu_1 = J - 1$ and $\nu_2 = IJ(K-1)$ degrees of freedom, and can be used to test H_B.

Similarly, the hypothesis H_{AB}: all $\gamma_{ij} = 0$ yields

$$SS_{\min_{AB}} SS_e + K \sum_i \sum_j (y_{ij.} - y_{i..} - y_{.j.} + y_{...})^2.$$

We define

$$SS_{AB} = SS_{\min_{AB}} - SS_e$$

$$= K \sum_i \sum_j (y_{ij.} - y_{i..} - y_{.j.} + y_{...})^2. \tag{70}$$

If normality is assumed, SS_{AB} has a chi-square distribution with $(I-1)(J-1)$ degrees of freedom and

$$F_{AB} = \frac{SS_{AB}/(I-1)(J-1)}{SS_e/IJ(K-1)} \tag{71}$$

has an F distribution with $\nu_1 = (I-1)(J-1)$ and $\nu_2 = IJ(K-1)$ degrees of freedom. If $F_{AB} > F_{\alpha;\nu_1,\nu_2}$, we reject the hypothesis H_{AB} at a level of confidence (false-alarm probability) of α.

Tests of these hypotheses without the assumption of normality will be discussed later.

Significance of Effects Revealed by the Data

If one or more of the hypotheses (63) are rejected, one would naturally wish to determine which effect or which interaction led to the rejection. The usual "t" test of data that appear large is not valid unless the experiment was designed to test the particular hypothesis suggested by the data. Valid statistical tests of multiple comparisons have been derived by Tukey and by Scheffé [47, 60, 61]. Tukey's method can be applied to the row (wavenumber) and to the column (frequency) effects but not to the interactions, since it requires equal covariances.

Scheffé's test is specifically applicable to "contrasts" suggested by the way the data fall out. A contrast among the parameters $\alpha_1, \ldots, \alpha_I$ is a linear function of these parameters.

$$\psi = \sum_{i=1}^{I} c_i \alpha_i ,$$

where the c_i are known constant coefficients subject to the condition

$$\sum_{c=1}^{I} c_i = 0 .$$

A more useful contrast than the difference $\alpha_i - \alpha_j$ between any two parameters is the difference between the averages of any two subsets of the parameters. The interactions themselves are contrasts $(\gamma_{ij} = \eta_{ij} - \eta_{i.} - \eta_{.j} + \eta_{..})$. We shall be concerned with these as well as contrasts among interactions, e.g., between the interaction of one cell and the average interactions of its neighbors.

An estimate of a contrast in interactions is

$$\hat{\psi} = \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} \hat{\gamma}_{ij}$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} (y_{ij.} - y_{i..} - y_{.j.}), \tag{72}$$

using (52), where

$$\sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} = 0.$$

In general, $\hat{\psi} \neq 0$. We say that $\hat{\psi}$ is significantly different from zero if and only if

$$|\hat{\psi}| > S\hat{\sigma}_{\hat{\psi}}, \tag{73}$$

where $S^2 = \nu_1 F_{\alpha;\, \nu_1,\, \nu_2}$, $\nu_1 = (I-1)(J-1)$, $\nu_2 = IJ(K-1)$, and $\hat{\sigma}^2 \hat{\psi}$ is an estimate of var $\hat{\psi}$ given by

$$\hat{\sigma}^2 \hat{\psi} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{i'=1}^{I} \sum_{j'=1}^{J} c_{ij} c_{i'j'} \operatorname{cov}(\hat{\gamma}_{ij}, \hat{\gamma}_{i'j'})$$

$$= s^2 \quad \frac{1}{K} \sum_i \sum_j c_{ij}^2 - \frac{1}{JK} \sum_i \sum_j \sum_{j'} c_{ij} c_{ij'} - \frac{1}{IK} \sum_i \sum_{i'} \sum_j c_{ij} c_{i'j}). \tag{74}$$

Here s^2 is an unbiased estimate of the variance σ^2 and is given by

$$S^2 = SS_e / IJ(K-1), \tag{75}$$

where SS_e is given by Eq. (64).

The F test based on Eq. (71) will lead to a rejection of the hypothesis H_{AB} if and only if some contrast in the interactions is significantly different from zero. The F test can therefore be considered as a preliminary search to determine the existence of a target (detection), with the test of contrasts used to determine its parameters (estimation). Tests of significance of contrasts in the row and column effects are similarly defined.

Interpretation of Results

It may happen that the hypothesis of no interactions is rejected and the hypotheses of no wavenumber and no frequency effects are both accepted. In this case we conclude that there must be differences in these effects but that the data are insufficient to reveal these differences when the wavenumber effects are averaged over all frequencies and vice versa. The interactions are of primary importance in our application.

If the hypothesis of no frequency effects is rejected and the other hypotheses are accepted, we may suspect the existence of isotropic, single-frequency noise.

A broadband source in the direction of the axis of the array may result in a significant effect at zero wavenumber with no frequency effect or interaction. Our definitions imply that no other spatial direction corresponds to a single wavenumber, so that a physical process giving rise to other row effects without column effects and interactions does not appear meaningful for acoustic data transformed from space-time to wavenumber-frequency.

Extension to Nonnormal and Nonwhite Data

As mentioned previously the least-squares estimates of the interactions, Eq. (62), are valid for any distribution of the data. This is illustrated in computer-simulated experiments discussed below. For nonnormal data, of course, the statistic defined by Eq. (71) no longer has an F distribution. Unlike the usual analysis-of-variance application, however, the number of degrees of freedom ($\nu_1 = (I-1)(J-1)$, $\nu_2 = IJ(K-1)$) is quite large in our array application. If the distribution of the data is known, some convenient function of the statistic defined by Eq. (71) or Eq. (73) may be asympotically normal, as is $z = (\ell n\, F)/2$. In any case, an empirically determined threshold can suffice to determine significance.

Simulated Experiments

The above analysis has been applied to computer-simulated sinusoidal signals in normal and nonnormal noise backgrounds. If the noise is normally distributed in the space-time domain, its amplitude is normally distributed in the wavenumber-frequency domain, its power spectrum has a chi-square distribution, with a Rayleight distribution of magnitude and a uniform distribution of phase. Our program permits analysis of all four of these distributions generated from the same "data" samples.

Figure 1 is a computer printout of the analysis of the amplitude spectrum of a sinusoidal signal in white normal noise. In this case the signal-to-noise ratio was -6 dB, and eight replications on an eight-by-32 array were simulated. The "target" was placed at a point corresponding to about 1/10 of the distance between the 23rd and 24th column and 3/4 of the distance between the 5th and 6th rows. The "data" did not reveal a significant row effect, but the column and interaction effects are highly significant. The five largest of each category are printed in Fig. 1. Note that even though the row effect is lost, the row-column interaction appears as expected. Here the real and imaginary parts of the spectrum are treated in adjacent columns.

Figure 2 shows the results of the analysis of the power spectrum of the same "data." Since there are now only half as many columns, the degrees of freedom are not the same as those of Fig. 1, and the "target" frequency corresponds to column 12. The statistics labeled F, of course, have the F distribution only when the transforms of the observations are normally distributed and not in this case (chi square) or in the cases to follow in Figs. 3 and 4.

```
RUN

DATA FILE NAME=? /DN1/
CONTRAST FILE NAME=? /CN1/
NO. LARGEST: ROW,COL,INT=? 5,5,5


 F(A) =        .405, D.F.=       7,     1792
 F(B) =       2.715, D.F.=      31,     1792
 F(AB)=       2.860, D.F.=     217,     1792


LARGEST ROW EFFECTS

    R(     7)=       1.114
    R(     1)=        .652
    R(     8)=        .117
    R(     2)=        .077
    R(     3)=       -.183

LARGEST COL EFFECTS

    C(    23)=      14.367
    C(    24)=       4.310
    C(    26)=       3.203
    C(     3)=       3.136
    C(     7)=       2.584

LARGEST INTERACTIONS

    T(     6, 23)=        85.916
    T(     5, 24)=        24.519
    T(     6, 17)=        14.476
    T(     5, 25)=        13.236
    T(     4, 24)=        12.858
```

Fig. 1 - Computer printout of the analysis of variance of the amplitude spectrum of a signal in normally distributed noise for a signal-to-noise ratio of -6 dB

Figures 3 and 4 show the analysis of the Rayleigh-distributed magnitude and the uniform-distributed phase, respecively, of the transforms of the same "data." Had the "target" been placed on a point corresponding to one cell, the phase of the signal would have been zero, and the analysis of Fig. 4 would be meaningless. Thus the value of the phase is not a very meaningful test. Nevertheless it is interesting to note that the "correct" cell does have the largest interaction in Fig. 4. Note also that the noise cells among the "top five" differ in these analyses.

The signal-to-noise ratio of the Fig. 5 "data" was -20 dB, the "target" corresponds to cell (3, 17), and all else is as in Fig. 1. The row effect, the column effect, and the interactions are all not significant. The interactions were statistically significant in three out of ten simulated experiments at this level; however the largest interaction corresponded to the "correct" cell in all ten. The largest interaction indicated the "correct" cell in six out of ten simulated experiments at the -23-dB level; two of these showed statistically significant interaction effects. At the -26-dB level no interaction effects were significant, and the "correct" cell was indicated in two out of ten simulated experiments.

For a known distribution of observations the significance points of the distributions may be calculated. If the distribution is unknown but can be assumed stationary, it may be possible to obtain empirical significance values.

```
RUN

DATA FILE NAME=? /DC1/
CONTRAST FILE NAME=? /CC1/
NO. LARGEST: ROW,COL,INT=? 5,5,5


 F(A) =      42.433, D.F.=      7,      896
 F(B) =      46.832, D.F.=     15,      896
 F(AB)=      30.517, D.F.=    105,      896



LARGEST ROW EFFECTS

    R(      6)=     835.465
    R(      5)=     -57.847
    R(      2)=     -61.006
    R(      8)=     -90.859
    R(      1)=    -112.001

LARGEST COL EFFECTS

    C(     12)=    1884.834
    C(     11)=      27.136
    C(      9)=      -5.966
    C(      3)=     -58.687
    C(      7)=     -98.387

LARGEST INTERACTIONS

    T(      6, 12)=        10617.130
    T(      7, 10)=          540.644
    T(      1, 10)=          503.260
    T(      4,  9)=          502.473
    T(      7,  7)=          496.408
```

Fig. 2 - Computer printout of the analysis of variance of the power spectrum of a signal in normally distributed noise for a signal-to-noise ratio of -6 dB

Application to observations on phase stability rather than phase itself may be more meaningful, since the latter may be zero in cases of interest.

It is possible to extend this analysis to Fourier transforms of four-dimensional space-time samples. Let

$$y_{ijk\ell m} = \eta_{ijk\ell} + e_{ijk\ell m}$$

represent the m^{th} observation of the i^{th}, j^{th}, and k^{th} components of the wavenumber along the x, y, and z axes respectively at the ℓ^{th} frequency, where the $\{\eta_{ijk\ell}\}$ are unknown constants and the $\{\eta_{ijk\ell_m}\}$ are random variables. Of primary interest here is not the separate wavenumber and frequency effects but the four-way interaction estimated by

$$\begin{aligned}\hat{\alpha}^{ABCD}_{ijk\ell} &= y_{ijk\ell.} - y_{ijk..} - y_{ij.\ell.} - y_{i.k\ell.} - y_{.jk\ell.} \\ &+ y_{ij...} + y_{i.k..} + y_{.jk..} + y_{i..\ell.} + y_{.j.\ell.} + y_{..k\ell.} \\ &- y_{i....} - y_{.j...} - y_{..k..} - y_{...\ell.} + y_{.....},\end{aligned}$$

```
RUN

DATA FILE NAME=? /DR1/
CONTRAST FILE NAME=? /CR1/
NO. LARGEST: ROW,COL,INT=? 5,5,5


 F(A) =      11.994, D.F.=      7,      896
 F(B) =      13.578, D.F.=     15,      896
 F(AB)=       5.459, D.F.=    105,      896



LARGEST ROW EFFECTS

    R(     6)=      7.410
    R(     2)=       .021
    R(     8)=      -.145
    R(     5)=      -.316
    R(     4)=      -.941

LARGEST COL EFFECTS

    C(    12)=     16.893
    C(    11)=      1.968
    C(     9)=      1.343
    C(     3)=      -.085
    C(     7)=      -.302

LARGEST INTERACTIONS

    T(     6,  12)=          71.562
    T(     4,   9)=           8.426
    T(     7,  10)=           8.261
    T(     7,   7)=           6.985
    T(     2,   6)=           6.914
```

Fig. 3 - Computer printout of analysis of variance of the magnitude spectrum of a signal in normally distrubuted noise for a signal-to-noise ratio of -6 dB

and tests of the hypotheses H_{ABCD}: all $\alpha_{ijk\ell}^{ABCD} = 0$.

Under assumptions of normality, statistical independence, and equal variances the statistic

$$F_{ABCD} = \frac{M \sum_i \sum_j \sum_k \sum_\ell \left(\hat{\alpha}_{ijk\ell}^{ABCD}\right)^2 \Big/ (I-1)(J-1)(K-1)(L-1)}{\sum_i \sum_j \sum_k \sum_\ell \sum_m (y_{ijk\ell m} - y_{ijk\ell .})^2 \Big/ IJKL(M-1)}$$

has an F distribution with $\nu_1 = (I-1)(J-1)(K-1)(L-1)$ and $\nu_2 = IJKL(M-1)$ degrees of freedom [47]. One obvious application would be to sampled data from a three-dimensional acoustic array.

```
RUN

DATA FILE NAME=? /DU1/
CONTRAST FILE NAME=? /CU1/
NO. LARGEST: ROW,COL,INT=? 5,5,5


 F(A) =        .345, D.F.=       7,      896
 F(B) =       1.428, D.F.=      15,      896
 F(AB)=       1.020, D.F.=     105,      896


LARGEST ROW EFFECTS

    R(      6)=       .131
    R(      3)=       .101
    R(      8)=       .073
    R(      2)=      -.005
    R(      1)=      -.025

LARGEST COL EFFECTS

    C(     15)=       .463
    C(      8)=       .359
    C(      3)=       .267
    C(      4)=       .219
    C(     10)=       .132

LARGEST INTERACTIONS

    T(      6, 12)=          3.114
    T(      1, 15)=          1.337
    T(      7, 16)=          1.110
    T(      7, 10)=           .828
    T(      3,  4)=           .824
```

Fig. 4 - Computer printout of the analysis of variance of the phase spectrum of a signal in normally distributed noise for a signal-to-noise ratio of -6 dB

```
RUN

DATA FILE NAME=? /DN1/
CONTRAST FILE NAME=? /CN1/
NO. LARGEST: ROW,COL,INT=? 5,5,5


 F(A) =        1.410, D.F.=      7,     1792
 F(B) =         .972, D.F.=     31,     1792
 F(AB)=        1.081, D.F.=    217,     1792



LARGEST ROW EFFECTS

      R(     3)=      7.236
      R(     4)=      5.499
      R(     1)=      5.119
      R(     7)=      -.700
      R(     6)=      -.930

LARGEST COL EFFECTS

      C(     6)=     17.265
      C(    17)=     15.315
      C(    26)=     13.379
      C(    15)=     12.783
      C(    24)=     12.486

LARGEST INTERACTIONS

      T(     3,  17)=         88.129
      T(     6,  12)=         76.018
      T(     3,  25)=         73.196
      T(     5,   1)=         69.692
      T(     3,  18)=         55.360
```

Fig. 5 – Computer printout of the analysis of variance of the amplitude spectrum of a signal in normally distributed noise for a signal-to-noise ratio of –20 dB

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Appendix A

SAMPLING THEORY

E. T. Whittaker [A1], J. M. Whittaker [A2], Shannon [A3], and others have shown that a function $f(x)$ whose Fourier transform $g(y) \in L_2(-a, a)$ and vanishes outside $(-a, a)$ can be exactly represented by

$$f(x) = \sum_{-\infty}^{\infty} f(n/2a)\ [\sin \pi(2ax - n)]/\pi(2ax - n). \tag{A1}$$

Thus if $f(x)$ is to be estimated by sampling, the set $\{f(n/2a)\}$ is a sufficient statistic. Furthermore, if T is any 1:1 transformation, then $T\{f(n/2a)\}$ is also a sufficient statistic. This concept of "conservation of information" justifies mapping the $\{f(n/2a)\}$ into any space which facilitates analysis. Given the $\{f(n/2a)\}$ the generality of transformations available for analysis is limited solely by the available computational facility.

In a practical application of this theorem the band-limited constraint of f does not generally create any serious difficulties; the spacing $1/2a$ of the uniform samples may be difficult to achieve; and of course the infinite set is never available. On the other hand the exactitude provided by Eq. (A1) is seldom required in practice. The explicit determination of f for all x is also seldom required. Errors introduced by improper spacing and inadequate sampling may be amenable to analysis.

An array may be considered to provide a set of multidimensional samples, discrete in one, two, or three spatial dimensions and either continuous or discrete in time. The multidimensional analogs of the sampling theorems discussed here will therefore permit a very general analysis of the information sampled by the array.

ALIASING

A set of equally spaced samples, $\{f(n/2a)\}$ does not uniquely determine a function but rather a set of functions: Whittaker's cotabular set (the set of aliases [A4]). Equation (A1) determines the cardinal function (the principal alias) - the unique member of the cotabular set with the smallest maximum frequency component $\leq a$, where $a = 1/2\Delta x$ is the Nyquist (folding, or cutoff) frequency, with Δx being the (uniform) sample spacing. Thus if a function $g(x)$ which contains a component at frequency $\nu > a$ is sampled with a sample spacing of $\Delta x = 1/2a$, that component will be replaced by its principal alias in the reconstruction by Eq. (A1).

POISSON SAMPLING

The problem of aliasing is an inevitable consequence of equally spaced sampling. No workable scheme seems so far to have been developed to sample at a definite but not uniformly spaced pattern and thus avoid aliasing [A4,A5]. We will comment further on this possibility in a later section. One approach to alias-free sampling is in the work of

Shapiro and Silverman, [A5], who show that if the sample points are randomly distributed in accordance with the Poisson probability law, the sampling is alias free. If there are μ samples per unit time (distance), then the probability of exactly n samples in a period of length T is

$$p_n(\mu T) = e^{-\mu T}(\mu T)^n/n! \tag{A2}$$

The waiting time (distance) to the next sample then has the exponential probability density function $\mu e^{-\mu x}$. That Poisson sampling is alias free "follows ... from the completeness of the Laguerre functions" [A5].

There are some interesting transformations related to Poisson sampling. Bolgiano and Piovoso [A6, A7] claim an efficient representation of certain waveforms is obtained by a "Poisson transformation." This transformation may be defined as

$$f_n = \int_0^\infty f(x)p_n(x)\,dx, \tag{A3}$$

where p_n is given by Eq. (A2) and

$$f(x) = \sum_{n=0}^{\infty} f_n g_n(x), \tag{A4}$$

in which

$$g_n(x) = (-1)^n\, 2^{n+1}\, e^{-x} \sum_{\nu=0}^{\infty} \binom{n+\nu}{n} L_{n+\nu}(2x),$$

where

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} (-x)^k/k!$$

are the Laguerre polynomials [A8]. As with the efficient representations by Huggins [A9] and others in terms of orthonormalized exponentials, this "Poisson transform," Eq. (A3) cannot be directly applied to discrete (sampled) data.

A related discrete transform (essentially a Gram-Charlier series, Type B) is given by Schmidt [A10]. Letting $\lambda = \mu T$ in Eq. (A2) we can write

$$F_k(\lambda) = \frac{\lambda^k}{k!} \sum_{n=0}^{\infty} f(n)\; p_n^{(k)}(\lambda), \tag{A5}$$

where

$$f(n) = p_n(\lambda) \sum_{k=0}^{\infty} F_k(\lambda)\; p_n^{(k)}(\lambda),$$

and

$$p_n^{(k)}(\lambda) = \frac{1}{p_n(\lambda)} \frac{d^k p_n(\lambda)}{d\lambda^k}$$

(Expressed in terms of the associated Laguerre polynomials [A8],

$$L_n^{(\alpha)}(x) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \frac{(-x)^\nu}{\nu!},$$

$p_n^{(k)}(\lambda)$ becomes

$$p_n^{(k)}(\lambda) = k!\lambda^{-k} L_k^{(n-k)}(\lambda) = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu}\binom{n}{\nu} \nu!\lambda^{-\nu}.$$

The relationship of the transformation given by Eq. (A5) to the "efficient" representation given by Eq. (A3) and to alias-free sampling remains to be shown. The generalizations of the sampling theorem discussed below may provide a useful connection, with a sampling theorem based on a Poisson-Laguerre type transformation.

GENERALIZATIONS OF THE SAMPLING THEOREM

Weiss [A11] and Kramer [A12] have generalized the sampling theorem to functions which satisfy integral transformations other than the usual Fourier one. Let $f(x)$ satisfy a Fredholm equation of the first kind:

$$f(x) = \int_I K(x, y) g(y) dy, \tag{A7}$$

in which the kernel $K(x, y) \in L_2(I)$ for each real x, $g(y) \in L_2(I)$, and there exists a countable set $\{x_j\}$ such that $\{K(x_j, y)\}$ is a complete orthogonal set on $L_2(I)$. Then

$$f(x) = \sum_{-\infty}^{\infty} f(x_j)\, I(x, x_j) \tag{A8}$$

is a generalization of Eq. (A1). The interpolation functions are given by

$$I(x, x_j) = \frac{\int_I K(x, y)\overline{K(x_j, y)}\, dy}{\int_I |K(x_j, y)|^2 dy}. \tag{A9}$$

Campbell [A13] and Jerri [A14,A15] have compared the WKS (after both Whittakers [A1,A2] , Kotelnikov [A16], and Shannon [A3]) sampling theorem, Eq. (A1), with the Kramer generalization, the WKSK sampling theorem, Eq. (A8). They conclude that the classes of functions samplable with each of these theorems are identical. Jerri [A15] states that the "possible advantage of the WKSK sampling theorem" may become clear when one "considers other integral transforms besides the Fourier one" for analysis.

Another possible advantage may be conjectured by noting that the class of sets of which $\{x_j\}$ is an element is a broad one with few restrictions. In at least some cases it may be possible to find a complete orthogonal set $\{K(x_j, y)\}$ corresponding to a given $\{x_j\}$, *equally spaced or not*. A sampling theorem using such a set may avoid the aliasing problem.

Following the work of Jerri [A14], Kramer-type sampling theorems based on Legendre functions and on Bessel functions have been derived. Using transformations based on Legendre polynomials, three sampling theorems (with unit sampling interval $a = 1/2$) are

$$f(x) = \sum_{j=0}^{\infty} f(j) \frac{\sin \pi(x-j)}{\pi(x-j)} \left(\frac{2j+1}{x+j+1}\right), \quad x \geq 0, \tag{A10}$$

$$f(x) = \sum_{j=0}^{\infty} f\left(j+\frac{1}{2}\right) \frac{\sin \pi\left(x-j-\frac{1}{2}\right)}{\pi\left(x-j-\frac{1}{2}\right)} \left(\frac{2j+1}{x+j+\frac{1}{2}}\right), x \geq 0, \tag{A11}$$

and

$$f(x) = \sum_{j=1}^{\infty} f(j) \frac{\sin \pi(x-j)}{\pi(x-j)} \left(\frac{2j-1}{x+j-1}\right), \quad x \geq 0. \tag{A12}$$

The second of these, Eq. (A11), is given by Campbell [A13] and by Jerri [A14]. A striking feature of these theorems is their similarity to each other and to Eq. (A1). Yet they are not identical and produce different results in approximations using a small number of terms.

Using the transformation

$$f(x) = \int_0^1 \sqrt{y}\, J_n(y\sqrt{x})\, g(y)\, dy$$

we obtain the Bessel-function sampling theorem

$$f(x) \sum_{j=1}^{\infty} f(x_j) \frac{2\sqrt{x_j}\, J_n\left(\sqrt{x}\right)}{(x_j - x)\, J_{n+1}\left(\sqrt{x_j}\right)}, \tag{A13}$$

where

$$J_n\left(\sqrt{x_j}\right) = 0, \quad j = 1, 2, \ldots .$$

This sampling theorem was obtained by Kramer [A12] and by Jerri [A14].

The transformation

$$f(x) = \int_0^1 y J_n(xy) g(y)\, dy$$

yields an alternative Bessel-function sampling theorem given by Campbell [A13]:

$$f(x) = \sum_{j=1}^{\infty} f(x_j) \frac{2x_j J_n(x)}{\left(x_j^2 - x^2\right) J_{n+1}(x_j)}, \tag{A14}$$

where

$$J_n(x_j) = 0, \quad j = 1, 2, \ldots .$$

Note that in Eq. (A14) the sample points x_j are asymptotically equally spaced, whereas in Eq. (A13) they have unequal spacing.

STOCHASTIC PROCESSES

Sampling theorems have also been derived for stochastic processes [A17, A18]. The set $\{f(t_j)\}$ are then regarded as "the observed values of a multivariate complex. Their characteristic feature, however, is that the order of the set $t_1, t_2, \ldots, t_n$ is material and not, for example, accidental as it would be for a random sample $x_1, x_2, \ldots, x_n$, in which the suffixes are adjoined for convenience of identification" [A19]. Recent work by Shumway and Dean [A20] and by Shumway [A21] indicates methods of application of statistical-estimation, regression, and analysis-of-variance techniques to these problems.

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Appendix B

INTERPOLATION USING FINITE FOURIER SERIES

In the section "Interpolations and Transformations" $f(x)$ is a complex function of a multidimensional vector. We now consider use of finite Fourier series to interpolate data sampled from a real continuous function $f(x)$ of the single independent variable.

If an odd number of sample points $(2n+1)$ is located within an interval L of x, $f(x)$ may be represented by

$$f(x) = (a_0/2) + \sum_{k=1}^{n} \left[a_k \cos(2\pi kx/L) + b_k \sin(2\pi kx/L)\right]. \tag{B1}$$

If the number of points is even $(2n)$, one representation is

$$f(x) = \sum_{k=0}^{n} \left[a_k \cos(2\pi kx/L + b_k \sin(2\pi kx/L)\right] \tag{B2}$$

or alternatively

$$f(x) = \sum_{k=1}^{n} \left\{a_k \cos[\pi(2k-1)x/L] + b_k \sin[\pi(2k-1)x/L]\right\}. \tag{B3}$$

As is discussed in the Refs. B1 through B4 the equations are derived from the usual complex Fourier representation by imposing the condition that $f(x)$ be real for all x. These equations simply represent possible choices of basis functions for the approximation of $f(x)$ over the interval. When the sample points are equally spaced on the interval $(x = (j-1)d$, where d is a constant), the basis functions are orthogonal on the sampling points, and the coefficients may readily be computed. This is the primary reason for the widespread use of the above representations. For example the coefficients for Eq. (B1) are easily shown to be

$$a_k = \frac{2}{2n+1} \sum_{j=0}^{2n+1} f(x_j) \cos\left[2\pi k(j-1)/(2n+1)\right] \tag{B4}$$

and

$$b_k = \frac{2}{2n+1} \sum_{j=1}^{2n+1} f(x_j) \sin\left[2\pi k(j-1)/(2n+1)\right]. \tag{B5}$$

Substituting these into Eq. (B1) yields

$$f(x) = \frac{1}{2n+1}\left(\sum_{j=1}^{2n+1} f(x_j) + 2\sum_{k=1}^{n}\sum_{j=1}^{2n+1} f(x_j)\left\{\cos(2\pi kx/L)\cos[2\pi k(j-1)/(2n+1)] + \sin(2\pi kx/L)\sin[2\pi k(j-1)/(2n+1)]\right\}\right)$$

which, with $d = L/(2n+1)$, becomes

$$f(x) = \sum_{j=1}^{2n+1} f(x_j)\,\frac{1}{2n+1}\left\{1 + 2\sum_{k=1}^{n}\cos\left[2\pi k\left(\frac{x}{d} - j + 1\right)/(2n+1)\right]\right\}.$$

This is of the form of (16) with the interpolation function

$$I(x, x_j) = \frac{1}{2n+1}\left\{1 + 2\sum_{k=1}^{n}\cos\left[2\pi k\left(\frac{x}{d} - j + 1\right)/(2n+1)\right]\right\}. \quad \textbf{(B6)}$$

The corresponding interpolation function derived from Eq. (B3) is

$$I(x, x_j) = \frac{1}{n}\sum_{k=1}^{n}\cos\left[\pi(2k-1)\left(\frac{x}{d} - j + 1\right)/2n\right], \quad \textbf{(B7)}$$

and that corresponding to Eq. (B2) is

$$I(x, x_j) = \frac{1}{2n}\left\{1 + 2\sum_{k-1}^{n-1}\cos\left[2\pi k\left(\frac{x}{d} - j + 1\right)/2n\right] + \cos\left[2\pi n\left(\frac{x}{d} - j + 1\right)/2n\right]\right\}.$$

As the number of terms tends toward infinity, (B6), (B7), and (B8) tend toward the same limit,

$$I(x, x_j) \to \frac{\sin\pi(x/d - j + 1)}{\pi(x/d - j + 1)}, \quad \textbf{(B9)}$$

which is equivalent to the usual interpolation function of Eq. (A1). Although Eqs. (B7) and (B8) are interpolation functions for use on the same data, they are different, a fact based on the different choice of basis functions in Eqs. (B2) and (B3). This is analogous to the interpolation functions of (A1), (A10), A11), and to (A12). The specific choice of basis functions which will be "best" will depend on the data.

Numerical examples of $I(x, x_1)$ for $d = 1$ are tabulated in Table B1 ($n = 10$) and Table B2 ($n = 100$), where S1, S2, S3, and S4 correspond to Eqs. (B9), (B6), (B7) and (B8) respectively. Note that, for the same number of terms, the difference between Eq. (B8) (S4) and Eq. (B9)(S1) is about twice that of Eq. (B7)(S3) and Eq. (B9)(S1); Eqs. (B6) and (B7) yield very similar results. All of these functions obey the necessary condition for interpolation functions (with linearly independent sample values):

$$I(x_k, x_j) = \delta_{kj}. \quad \textbf{(B10)}$$

The procedure followed above in deriving the interpolation functions is based on availability of closed-form expressions (e.g., Eqs. (B4)and (B5) for the coefficients in (1).

The matrices defined prior to Eq. (17) may be used directly to compute this equation. For the case of interpolation, namely $m = n$, $U = P^{-1}$. As an illustration of this direct approach, consider the approximation in Eq. (B2), based on an even number of sample points ($2n$).

The basis functions may be written as

$$S_k(x) = \begin{cases} \cos\ [2\pi(k-1)x/L], & k = 1, 2, \ldots, n+1, \\ \sin\ [2\pi(k-n-1)x/L], & k = n+2, n+3, \ldots, 2n. \end{cases} \tag{B11}$$

Evaluation of Eq. (B11) at the points x_j gives the matrix P. If this matrix is nonsingular, its inverse P^{-1} may be computed and substituted into Eq. (17) to obtain the desired interpolation function. Note that this procedure does not depend on the orthogonality of the basis functions, on the x_j chosen, or on any particular choice of the x_j .

A computer program which carries out the procedure outlined above and prints the interpolation functions over a range of x for $j = 1, 2, \ldots, 5$ has been written. The sample points x_j may be equally or unequally spaced. Some results are shown in Table B3 (equally spaced x_j) and Table B4 (unequally spaced x_j). We let $2n = 20$, to correspond with the tabulation in Table B1. (Compare S4 of Table B1 with the case $j = 1$ of Table B3.) Note that, as indicated in Eq. (B8), the interpolation functions are symmetrical about $x = x_j$ and the form of the functions is independent of j, that is, all of the tabulations in Table B3 are of the same function simply displaced with respect to the tabulations for other values of j . The results in Table B3 satisfy Eq. (B10).

The tabulations for unequally spaced x_j in Table B4 are also seen to satisfy Eq. (B10), although the functions are not symmetric about $x = x_j$ and the form of the interpolation functions depends on j. It should be noted that some of the functions assume values greater than unity for some values of x, which can occur when interpolation is carried out on data taken at unequally spaced intervals.

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Table B1
Interpolation Functions (Eqs. (B6) through (B9) for $n = 10$)

X	S1	S2	S3	S4
0.0	1.000000	1.000000	1.000000	1.000000
.1	.983631	.983668	.983672	.983550
.2	.935489	.935628	.935643	.935181
.3	.858393	.858681	.858711	.857758
.4	.756826	.757278	.757324	.755830
.5	.636619	.637213	.637274	.635310
.6	.504551	.505229	.505298	.503056
.7	.367883	.368556	.368625	.366399
.8	.233872	.234431	.234488	.232639
.9	.109292	.109623	.109657	.108563
1.0	.000000	.000000	.000000	.000000
1.1	-.089421	-.089825	-.089867	-.088529
1.2	-.155914	-.156755	-.156842	-.154063
1.3	-.198090	-.199345	-.199474	-.195329
1.4	-.216236	-.217825	-.217989	-.212739
1.5	-.212206	-.213998	-.214182	-.208265
1.6	-.189206	-.191025	-.191213	-.185206
1.7	-.151481	-.153126	-.153296	-.147863
1.8	-.103943	-.105210	-.105341	-.101158
1.9	-.051770	-.052473	-.052546	-.050223
2.0	-.000000	-.000000	-.000000	-.000000
2.1	.046839	.047619	.047699	.045128
2.2	.085044	.086599	.086761	.081631
2.3	.111964	.114204	.114437	.107050
2.4	.126137	.128889	.129175	.120104
2.5	.127323	.130341	.130656	.120710
2.6	.116434	.119423	.119735	.109888
2.7	.095377	.098020	.098297	.089588
2.8	.066820	.068815	.069024	.062455
2.9	.033918	.035006	.035120	.031539
3.0	.000000	.000000	.000000	.000000
3.1	-.031730	-.032896	-.033019	-.029181
3.2	-.058468	-.060762	-.061004	-.053458
3.3	-.078035	-.081298	-.081643	-.070917
3.4	-.089038	-.092996	-.093416	-.080407
3.5	-.090945	-.095238	-.095694	-.081592
3.6	-.084091	-.088298	-.088746	-.074931
3.7	-.069599	-.073284	-.073677	-.061580
3.8	-.049236	-.051991	-.052286	-.043245
3.9	-.025221	-.026711	-.026870	-.021984
4.0	-.000000	-.000000	-.000000	-.000000
4.1	.023991	.025564	.025733	.020578
4.2	.044547	.047619	.047950	.037888
4.3	.059887	.064226	.064696	.050490
4.4	.068802	.074034	.074601	.057481
4.5	.070735	.076375	.076988	.058542
4.6	.065811	.071307	.071906	.053938
4.7	.054791	.059580	.060104	.044454
4.8	.038978	.042541	.042932	.031296
4.9	.020074	.021991	.022202	.015944
5.0	.000000	.000000	.000000	.000000

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Table B2
Interpolation Functions (Eqs. (B6) through (B9) for n = 100)

X	S1	S2	S3	S4
0.0	1.000000	1.000000	1.000000	1.000000
.1	.983631	.983632	.983632	.983630
.2	.935489	.935490	.935490	.935486
.3	.858393	.858396	.858396	.858387
.4	.756826	.756831	.756831	.756816
.5	.636619	.636626	.636626	.636606
.6	.504551	.504558	.504558	.504536
.7	.367883	.367890	.367890	.367868
.8	.233872	.233878	.233878	.233860
.9	.109292	.109296	.109296	.109285
1.0	.000000	-.000000	-.000000	-.000000
1.1	-.089421	-.089425	-.089425	-.089412
1.2	-.155914	-.155924	-.155924	-.155896
1.3	-.198090	-.198104	-.198104	-.198063
1.4	-.216236	-.216253	-.216253	-.216201
1.5	-.212206	-.212226	-.212226	-.212167
1.6	-.189206	-.189226	-.189226	-.189166
1.7	-.151481	-.151499	-.151499	-.151445
1.8	-.103943	-.103956	-.103957	-.103915
1.9	-.051770	-.051777	-.051777	-.051754
2.0	-.000000	-.000000	-.000000	-.000000
2.1	.046839	.046848	.046848	.046822
2.2	.085044	.085061	.085061	.085010
2.3	.111964	.111988	.111988	.111915
2.4	.126137	.126167	.126167	.126073
2.5	.127323	.127356	.127356	.127258
2.6	.116434	.116466	.116467	.116370
2.7	.095377	.095405	.095405	.095319
2.8	.066823	.066842	.066842	.066777
2.9	.033918	.033929	.033930	.033894
3.0	.000000	.000000	.000000	.000000
3.1	-.031730	-.031742	-.031742	-.031704
3.2	-.058468	-.058492	-.058492	-.058418
3.3	-.078035	-.078070	-.078070	-.077965
3.4	-.089038	-.089080	-.089080	-.088953
3.5	-.090945	-.090991	-.090991	-.090854
3.6	-.084091	-.084136	-.084136	-.084002
3.7	-.069599	-.069638	-.069638	-.069521
3.8	-.049236	-.049265	-.049265	-.049177
3.9	-.025221	-.025236	-.025237	-.025189
4.0	-.000000	-.000000	-.000000	-.000000
4.1	.023991	.024007	.024007	.023957
4.2	.044547	.044579	.044579	.044482
4.3	.059887	.059933	.059933	.059796
4.4	.068802	.068856	.068857	.068692
4.5	.070735	.070793	.070794	.070617
4.6	.065811	.065867	.065868	.065696
4.7	.054791	.054840	.054840	.054691
4.8	.038978	.039015	.039015	.038904
4.9	.020074	.020093	.020093	.020034
5.0	.000000	.000000	.000000	.000000

Table B3
Interpolation Function (Equally Spaced Sample Points)

X	J=1	J=2	J=3	J=4	J=5
0.00	1.000000	.000000	-.000000	-.000000	.000000
.10	.983550	.108563	-.050223	.031539	-.021984
.20	.935181	.232639	-.101158	.062455	-.043245
.30	.857758	.366399	-.147863	.089588	-.061580
.40	.755830	.503056	-.185206	.109888	-.074931
.50	.635310	.635310	-.208265	.120710	-.081592
.60	.503056	.755830	-.212739	.120104	-.080407
.70	.366399	.857758	-.195329	.107050	-.070917
.80	.232639	.935181	-.154063	.081631	-.053458
.90	.108563	.983550	-.088529	.045128	-.029181
1.00	.000000	1.000000	-.000000	-.000000	.000000
1.10	-.088529	.983550	.108563	-.050223	.031539
1.20	-.154063	.935181	.232639	-.101158	.062455
1.30	-.195329	.857758	.366399	-.147863	.089588
1.40	-.212739	.755830	.503056	-.185206	.109888
1.50	-.208265	.635310	.635310	-.208265	.120710
1.60	-.185206	.503056	.755830	-.212739	.120104
1.70	-.147863	.366399	.857758	-.195329	.107050
1.80	-.101158	.232639	.935181	-.154063	.081631
1.90	-.050223	.108563	.983550	-.088529	.045128
2.00	-.000000	.000000	1.000000	-.000000	.000000
2.10	.045128	-.088529	.983550	.108563	-.050223
2.20	.081631	-.154063	.935181	.232639	-.101158
2.30	.107050	-.195329	.857758	.366399	-.147863
2.40	.120104	-.212739	.755830	.503056	-.185206
2.50	.120710	-.208265	.635310	.635310	-.208265
2.60	.109888	-.185206	.503056	.755830	-.212739
2.70	.089588	-.147863	.366399	.857758	-.195329
2.80	.062455	-.101158	.232639	.935181	-.154063
2.90	.031539	-.050223	.108563	.983550	-.088529
3.00	.000000	-.000000	.000000	1.000000	-.000000
3.10	-.029181	.045128	-.088529	.983550	.108563
3.20	-.053458	.081631	-.154063	.935181	.232639
3.30	-.070917	.107050	-.195329	.857758	.366399
3.40	-.080407	.120104	-.212739	.755830	.503056
3.50	-.081592	.120710	-.208265	.635310	.635310
3.60	-.074931	.109888	-.185206	.503056	.755830
3.70	-.061580	.089588	-.147863	.366399	.857758
3.80	-.043245	.062455	-.101158	.232639	.935181
3.90	-.021984	.031539	-.050223	.108563	.983550
4.00	-.000000	.000000	-.000000	.000000	1.000000
4.10	.020578	-.029181	.045128	-.088529	.983550
4.20	.037888	-.053458	.081631	-.154063	.935181
4.30	.050490	-.070917	.107050	-.195329	.857758
4.40	.057481	-.080407	.120104	-.212739	.755830
4.50	.058542	-.081592	.120710	-.208265	.635310
4.60	.053938	-.074931	.109888	-.185206	.503056

Table B4
Interpolation Function (Unequally Spaced Sample Points)

X	J=1	J=2	J=3	J=4	J=5
0.00	1.000000	-.000000	.000000	-.000000	-.000000
.10	.821657	.327003	-.270840	.212656	-.146162
.20	.661960	.593622	-.460627	.355071	-.242033
.30	.521014	.802159	-.575647	.434888	-.293844
.40	.398541	.955949	-.623427	.460692	-.308383
.50	.293923	1.059175	-.612431	.441680	-.292727
.60	.206257	1.116691	-.551754	.387333	-.253996
.70	.134408	1.133834	-.450820	.307117	-.199121
.80	.077058	1.116249	-.319109	.210199	-.134637
.90	.032763	1.069717	-.165891	.105205	-.066512
1.00	-.000000	1.000000	.000000	-.000000	.000000
1.10	-.022789	.912696	.170371	-.098482	.060472
1.20	-.037152	.813122	.337833	-.184374	.111375
1.30	-.044591	.706199	.495935	-.252964	.150119
1.40	-.046530	.596376	.639252	-.300748	.175062
1.50	-.044290	.487562	.763448	-.325439	.185481
1.60	-.039067	.383083	.865295	-.325938	.181520
1.70	-.031919	.285660	.942662	-.302272	.164109
1.80	-.023755	.197404	.994471	-.255499	.134865
1.90	-.015334	.119831	1.020631	-.187594	.095969
2.00	-.007261	.053889	1.021945	-.101304	.050041
2.10	-.000000	-.000000	1.000000	-.000000	.000000
2.20	.006127	-.041886	.957039	.112492	-.051078
2.30	.010920	-.072232	.895834	.232053	-.100095
2.40	.014292	-.091843	.819540	.354439	-.144061
2.50	.016249	-.101799	.731563	.475438	-.180216
2.60	.016877	-.103382	.635417	.591021	-.206138
2.70	.016321	-.098007	.534608	.697471	-.219827
2.80	.014774	-.087155	.432506	.791499	-.219781
2.90	.012453	-.072310	.332251	.870337	-.205040
3.00	.009590	-.054907	.236664	.931811	-.175220
3.10	.006417	-.036283	.148177	.974383	-.130513
3.20	.003154	-.017636	.068782	.997183	-.071676
3.30	-.000000	-.000000	-.000000	1.000000	-.000000
3.40	-.002876	.015779	-.057133	.983266	.082749
3.50	-.005338	.029056	-.102063	.948013	.174392
3.60	-.007286	.039384	-.134699	.895814	.272414
3.70	-.008658	.046516	-.155379	.828703	.374062
3.80	-.009427	.050388	-.164823	.749097	.476433
3.90	-.009604	.051104	-.164079	.659693	.576576
4.00	-.009226	.048914	-.154460	.563374	.671590
4.10	-.008359	.044185	-.137473	.463104	.758718
4.20	-.007087	.037374	-.114756	.361830	.835438
4.30	-.005508	.029001	-.088003	.262388	.899538
4.40	-.003729	.019615	-.058902	.167416	.949186
4.50	-.001858	.009771	-.029069	.079278	.982982
4.60	-.000000	.000000	-.000000	.000000	1.000000